

Point massive particle in General Relativity

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Abstract

It is well known that the Schwarzschild solution describes the gravitational field outside compact spherically symmetric mass distribution in General Relativity. In particular, it describes the gravitational field of a point particle. Nevertheless, what is the exact solution of Einstein's equations with δ -type source is not known. In the present paper, we give mathematical meaning of nonlinear Einstein's equations with δ -type source and find asymptotically flat static spherically symmetric solution in the class of generalized functions of combined system of equations of motion for metric and point massive particle.

1 Introduction

In this article, we consider the classic problem: find the gravitational which is produced by a point massive particle. If particle is at rest, then the gravitational field is spherically symmetric. The spherically symmetric solution of the vacuum Einstein's equations is well known: it is the Schwarzschild solution [1]. Therefore, it is often stated that the Schwarzschild solution describes gravitational field of a point particle. This statement is incorrect because there is no δ -type energy-momentum tensor corresponding to a particle on the right hand side of Einstein's equations.

On the other hand, the solution of Einstein's equations outside point massive particle must be isometric to Schwarzschild solution. Therefore the natural question arises: "Where is the δ -function?". The answer turned out to be unexpected: δ -function corresponds to infinite value of the Schwarzschild radial coordinate. Namely, in this paper, we prove that the Schwarzschild solution in isotropic coordinates is the solution of Einstein's equations in a topologically trivial space-time \mathbb{R}^4 with δ -type source. The obtained space-time is geodesically complete.

Severe mathematical difficulties arise during solution of this problem. A solution of Einstein's equations must be understood in a generalized sense after integration with test functions because the δ -function stands on the right hand side. But there is no multiplication in the space of generalized functions and the question arises what is the mathematical

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meaning of the left hand side of Einstein's equations which are nonlinear. Besides, locally nonsummable functions arise during solution of equations, and some functionals must be attributed to them. In other words, regularization is needed. In the preset paper, the solution of Einstein's equations is found in a generalized sense: the equations are satisfied after integration with test functions. We choose the usual space $\mathcal{D}(\mathbb{R}^3)$ of infinitely differentiable functions on \mathbb{R}^3 with compact support as the space of test functions. Spherically symmetric solution is found in the conjugate space $\mathcal{D}'(\mathbb{R}^3)$ in which we use the analytic regularization for spherically symmetric functionals. Though no multiplication is defined in \mathcal{D}' , the left hand side of Einstein's equations is well defined.

The obtained solution turned out to be the well known Schwarzschild solution in isotropic coordinates. It appears after gluing together two exterior solutions of a black hole along the horizon. This solution is isometric to Einstein–Rosen bridge and is asymptotically flat not only at large distances but near the particle itself where the curvature of space-time tends also to zero. At large distances, the gravitational field is attractive. Attraction decreases near the particle and changes to repulsion after crossing the horizon. This repulsion results in geodesic completeness near the particle.

Attempts to interpret the Schwarzschild solution in terms of generalized functions were made earlier [2–6]. Our approach is essentially different.

2 The main equation

We consider three-dimensional Euclidean space \mathbb{R}^3 with Cartesian coordinates x^μ , $\mu = 1, 2, 3$, in which the rotational group $\mathbb{SO}(3)$ acts in the usual way. Let the positive definite Riemannian metric $g_{\mu\nu}(x)$ which defines Christoffel's symbols and the corresponding curvature be given on it. In our notations, the main geometric notions have the following form:

$$\Gamma_{\mu\nu}{}^\rho := \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}), \quad (1)$$

$$R_{\mu\nu\rho}{}^\sigma := \partial_\mu \Gamma_{\nu\rho}{}^\sigma - \partial_\nu \Gamma_{\mu\rho}{}^\sigma - \Gamma_{\mu\rho}{}^\epsilon \Gamma_{\nu\epsilon}{}^\sigma + \Gamma_{\nu\rho}{}^\epsilon \Gamma_{\mu\epsilon}{}^\sigma, \quad (2)$$

$$R_{\mu\nu} := R_{\mu\rho\nu}{}^\rho, \quad (3)$$

$$R := g^{\mu\nu} R_{\mu\nu}, \quad (4)$$

where $\Gamma_{\mu\nu}{}^\rho$ are Christoffel's symbols, $R_{\mu\nu\rho}{}^\sigma$ is the curvature tensor, $R_{\mu\nu}$ is the Ricci tensor, and R is the scalar curvature.

Consider the covariant equation

$$\sqrt{|g|}R = -16\pi M\delta(\mathbf{x}), \quad M = \text{const}, \quad (5)$$

where

$$\delta(\mathbf{x}) := \delta(x^1)\delta(x^2)\delta(x^3)$$

is the three-dimensional δ -function. The volume element $\sqrt{|g|}$ on the left hand side is needed for covariance because δ -function is the scalar density with respect to coordinate transformations.

We shall see in what follows that equation (5) is the main equation in the system of Einstein's and point particle equations. Moreover, this equation may have its own applications in differential geometry which are not discussed here.

We are seeking spherically symmetric solutions of equation (5), and that is in agreement with the symmetry of the right hand side. Any spherically symmetric metric is Weyl flat, which means that there exists the coordinate system where the metric has the form

$$g_{\mu\nu} = e^{2\phi} \delta_{\mu\nu},$$

where $\delta_{\mu\nu} := \text{diag}(+, +, +)$ is the Euclidean metric and $\phi = \phi(x)$. Then expressions for the Ricci tensor and scalar curvature follow from formulae (2)–(4):

$$\begin{aligned} R_{\mu\nu} &= \partial_\mu \partial_\nu \phi - \partial_\mu \phi \partial_\nu \phi + \delta_{\mu\nu} (\Delta \phi + \partial \phi^2), \\ e^{2\phi} R &= 4\Delta \phi + 2\partial \phi^2, \end{aligned} \quad (6)$$

where $\partial_\mu := \partial/\partial x^\mu$ is the partial derivative, $\Delta := \delta^{\mu\nu} \partial_\mu \partial_\nu := \partial^\mu \partial_\mu$ is the usual Laplacian in the Euclidean space \mathbb{R}^3 , and we introduced notation $\partial \phi^2 := \delta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$. Using the expression for the scalar curvature, the Ricci tensor can be written in the form

$$R_{\mu\nu} = \partial_\mu \partial_\nu \phi - \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} \delta_{\mu\nu} \partial \phi^2 + \frac{1}{4} g_{\mu\nu} R. \quad (7)$$

The main equation (5) for the Weyl flat spherically symmetric metric takes the form

$$4e^\phi \Delta \phi + 2e^\phi \partial \phi^2 = -16\pi M \delta(\mathbf{x}). \quad (8)$$

We rewrite this equation in another form which is suitable for the analysis. Introducing the new variable

$$f := e^\phi, \quad (9)$$

we obtain the following equation

$$\Delta f - \frac{\partial f^2}{2f} = -4\pi M \delta(\mathbf{x}). \quad (10)$$

Thus the main equation (5) for the Weyl Euclidean metric has the form (10). It is nonlinear because of the second term on the left hand side. Solution of this equation should be understood in the generalized sense after integration with a test function (see, i.e. [7]) because there is the δ -function on the right hand side. If we look for a solution in the space of functionals (generalized functions) \mathcal{D}' then serious problems arise. Since the equation is nonlinear, the generalized functions must be multiplied, but multiplication in \mathcal{D}' is absent. Second, we shall see in what follows that some terms in equation (10) are not locally summable, and hence the regularization of the appearing integrals is necessary. Though multiplication of functionals in \mathcal{D}' is not defined in general, for some particular functionals the left hand side has nevertheless definite meaning.

We look for spherically symmetric solution of equation (10) in the spherical coordinate system r, θ, φ . In the space of spherically symmetric functionals r^λ , $\lambda \in \mathbb{C}$, we use the analytic regularization [8]. Then the left hand side of the main equation (10) is well defined for these functionals.

Theorem 2.1. *A spherically symmetric asymptotically flat solution of equation (10) exists in the space of generalized functions \mathcal{D}' , for which the left hand side of equation (10) is defined. It has the form*

$$f = 1 + \frac{M}{r} + \frac{M^2}{4r^2} = \left(1 + \frac{M}{2r}\right)^2. \quad (11)$$

Proof. Simple calculations lead to the following expression for the second term on the left hand side of equation (10)

$$\frac{\partial f^2}{2f} = \frac{f'^2}{2f} = \frac{M^2}{2r^4}, \quad (12)$$

where prime denotes differentiation with respect to radius. Therefore equation (10) for the generalized function (11) takes the form

$$\Delta \left(1 + \frac{M}{r} + \frac{M^2}{4r^2} \right) - \frac{M^2}{2r^4} = -4\pi M \delta(\mathbf{x}).$$

The function M/r is the unique fundamental solution of the Laplace equation decreasing at infinity (see., for example, [7]):

$$\Delta \left(1 + \frac{M}{r} \right) = \Delta \frac{M}{r} = -4\pi M \delta(\mathbf{x}). \quad (13)$$

The function $M^2/2r^4$ is not locally summable and requires regularization. We use the analytic regularization for the generalized function r^λ , $\lambda \in \mathbb{C}$ [8]. For $\text{re } \lambda > -3$ the function r^λ is locally summable, and the functional

$$(r^\lambda, \varphi) := \int_{\mathbb{R}^3} dx r^\lambda \varphi$$

is defined for all test functions $\varphi \in \mathcal{D}(\mathbb{R}^3)$. This functional is analytically continued on the whole complex plane $\lambda \in \mathbb{C}$ except simple poles located on the real line $\lambda = -3, -5, -7, \dots$ [8]. The poles does not matter in the considered case because we do not get into them.

The equality

$$\Delta r^\lambda = \frac{1}{r^2} \partial_r (r^2 \partial_r r^\lambda) = \lambda(\lambda + 1) r^{\lambda-2}$$

is simply verified for $\text{re } \lambda > 2$. It remains valid also after analytic continuation to the point $\lambda = -2$. For $\lambda = -2$, it looks as follows

$$\Delta \frac{1}{r^2} = 2 \frac{1}{r^4}.$$

Thus, for analytic regularization, we attribute to the locally nonsummable function $1/r^4$ the functional

$$\left(\frac{1}{r^4}, \varphi \right) := \frac{1}{2} \left(\Delta \frac{1}{r^2}, \varphi \right) = \frac{1}{2} \left(\frac{1}{r^2}, \Delta \varphi \right).$$

The last functional is well defined.

We see that the nonlinear term on the left hand side of the equation (10) is canceled after integration with a test function

$$\Delta \frac{M^2}{4r^2} - \frac{M^2}{2r^4} = 0, \quad (14)$$

if the analytic regularization is used for the function $M^2/2r^4$. □

Equation (5) for a massive point particle was considered in [2–4] for Weyl Euclidean form of the metric. The authors used parameterization $f := \chi^2$ and obtain equation

$$\chi \Delta \chi = -2\pi M \delta(\mathbf{x}) \quad (15)$$

which is equivalent to equation (10). They proposed the solution

$$\chi(r) = 1 + \frac{M}{2r\chi(0)} \quad (16)$$

differing from that corresponding to solution (11) by the essential factor $\chi(0)$. This factor is obtained from the regularized δ -function and depends on small parameter ϵ ,

$$\chi(0) \sim \sqrt{\frac{M}{\epsilon}},$$

which is the “radius” of the regularized δ -function. In the limit $\epsilon \rightarrow 0$, solution (16) becomes trivial, $\chi = 1$, and satisfies equation $\sqrt{g}R = 0$ corresponding to a vacuum rather than point particle. So the solution proposed in [2–4] does not depend on mass M . A generalization of equation (5) for a charged massive particle interacting with electromagnetic field was also considered. In this case, the factor $\chi(0)$ becomes nontrivial, and the total mass of a particle is proportional to its charge. This effect is interpreted as the regularization of the self-energy of a point charge by gravitational interaction.

To show the difference in the approaches, let us consider equation (15) for

$$\chi = 1 + \frac{M}{2r}, \quad (17)$$

which is different from (16). Substitution of this solution into equation (15) yields

$$\left(1 + \frac{M}{2r}\right) \Delta \left(1 + \frac{M}{2r}\right) = \left(1 + \frac{M}{2r}\right) (-2\pi M) \delta(\mathbf{x}) = -2\pi M \delta(\mathbf{x}). \quad (18)$$

Expression $\frac{1}{r} \delta(\mathbf{x})$ is not defined. We define it as follows. Relation

$$\Delta \frac{r^\lambda}{r} = \lambda(\lambda - 1) r^{\lambda-3} + r^\lambda \Delta \frac{1}{r}$$

can be easily verified for $\text{re } \lambda > 3$. We analytically continue it to the point $\lambda = -1$ where it can be rewritten as

$$\frac{1}{r} \Delta \frac{1}{r} = \Delta \frac{1}{r^2} - \frac{2}{r^4}.$$

So, we define

$$\left(\frac{1}{r} \delta(\mathbf{x}), \varphi\right) := -\frac{1}{4\pi} \left(\frac{1}{r} \Delta \frac{1}{r}, \varphi\right) := -\frac{1}{4\pi} \left(\Delta \frac{1}{r^2} - \frac{2}{r^4}, \varphi\right).$$

For analytically regularized $\frac{1}{r^4}$ the right hand side is zero, and we get relation

$$\frac{1}{r} \delta(\mathbf{x}) = 0$$

which is valid in a generalized sense. In other words, the relation $r^\lambda \delta(\mathbf{x}) = 0$ which is valid for $\lambda > 0$ is analytically continued to the point $\lambda = -1$.

We see that solution (11) can be obtained either by solving equation (10) or (15). In both cases, we used the analytic regularization and get the same solution. Our solution (11) does depend on mass M . It is nontrivial for neutral particle and differs essentially from solution (16) considered in [2–4].

Thus we have found the generalized solution of equation (10) in the space of functionals $\mathcal{D}'(\mathbb{R}^3)$. For this solution the left hand side of the main equation is well defined. Unfortunately, we did not describe explicitly that subspace in $\mathcal{D}'(\mathbb{R}^3)$, for which the left hand side of the main equation is defined. This question is complicated and related to the uniqueness of the obtained solution. We leave it for future work.

3 Relation to the Schwarzschild solution

Consider four-dimensional topologically trivial space-time with coordinates t, ρ, θ, φ . We identify sections $t = \text{const}$ with three-dimensional space which is considered in spherical coordinate system ρ, θ, φ . Static spherically symmetric solution of vacuum Einstein's equations without δ -type sources is well known. It is called the Schwarzschild solution [1] and is usually written in the form

$$ds^2 = \left(1 - \frac{2M}{\rho}\right) dt^2 - \frac{d\rho^2}{1 - \frac{2M}{\rho}} - \rho^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (19)$$

The Schwarzschild solution is defined for $2M < \rho < \infty$ (outside the horizon $\rho_s = 2M$) and for $0 < \rho < 2M$ (under the horizon). It is asymptotically flat and tends to the Lorentz metric when $\rho \rightarrow \infty$. It is important that constant M entering the Schwarzschild solution is an integration constant and is not contained in the vacuum Einstein's equations at all. From mathematical point of view, constant M may take arbitrary values $M \in \mathbb{R}$. However, if we assume that the Schwarzschild solution describes the gravitational field outside a point particle, then comparison with the Newton gravitational law at large distances (see, i.e. [9]) tells us that the integration constant M is the mass of a particle and therefore must be positive.

The space part of the metric is not Weyl Euclidean, and hence the radial coordinate is denoted by the letter ρ . To compare it with the three-dimensional metric obtained in the previous section, we perform the coordinate transformation $r \mapsto \rho(r)$ and change the total sign of the metric to make it positive definite. Then the space part of the Schwarzschild metric (19) becomes

$$dl^2 = \frac{\rho'^2}{1 - \frac{2M}{\rho}} dr^2 + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (20)$$

To make the spatial part of the Schwarzschild metric Weyl Euclidean, the function $\rho(r)$ must satisfy the equation

$$\frac{\rho'^2}{1 - \frac{2M}{\rho}} = \frac{\rho^2}{r^2}.$$

For $\rho > 2M$, it reduces to equation

$$\frac{dr}{r} = \frac{d\rho}{\sqrt{\rho^2 - 2M\rho}},$$

which is simply integrated:

$$\ln(Cr) = \ln\left(\sqrt{\rho^2 - 2M\rho} + \rho - M\right),$$

where C is an integration constant. Thus the transformation to the new coordinate system is given by the function

$$Cr = \sqrt{\rho^2 - 2M\rho} + \rho - M. \quad (21)$$

The inverse transformation of the radial coordinate is

$$\rho = \frac{(Cr + M)^2}{2Cr}. \quad (22)$$

Now the spatial part of the Schwarzschild metric can be written in Weyl Euclidean form:

$$dl^2 = \frac{(Cr + M)^4}{4C^2r^4} [dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)].$$

We require this metric to be asymptotically Euclidean for $r \rightarrow \infty$. This requirement uniquely determines the integration constant: $C = 2$.

Finally, we obtain the following expression for the spatial part of the metric

$$dl^2 = \left(1 + \frac{M}{2r}\right)^4 [dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)]. \quad (23)$$

We have $g_{\mu\nu} = f^2\delta_{\mu\nu}$ in notations of the previous section. Therefore for the function f we get exactly expression (11).

It is easily calculated,

$$1 - \frac{2M}{\rho} = \left(\frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}}\right)^2.$$

Thus the Schwarzschild metric in isotropic coordinates has the form

$$ds^2 = \left(\frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}}\right)^2 dt^2 - \left(1 + \frac{M}{2r}\right)^4 [dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)]. \quad (24)$$

For $M > 0$, this metric is defined everywhere in \mathbb{R}^4 excluding the world line of the origin of the spherical coordinate system:

$$-\infty < t < \infty, \quad 0 < r < \infty, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi < 2\pi.$$

This metric satisfies vacuum Einstein's equations everywhere in \mathbb{R}^4 excluding the world line of the origin of the spherical coordinate system because it is obtained by the coordinate transformation (22) from the Schwarzschild metric. According to theorem 2.1, the space part of this metric is the fundamental solution of equation (5) with δ -type right hand side for all moments of time t .

Metric (24) is isotropic on all sections of the space-time $t = \text{const.}$ Therefore these coordinates for the Schwarzschild solution are called isotropic. They are well known for a long time, see, i.e. [9]. The new result is not the Schwarzschild metric in isotropic coordinates (24), but the statement that its spatial part satisfies equation (5) with δ -type source. This fact has principal meaning and far reaching physical consequences. We shall see in what follows that gravitational attraction to the point mass M at large distances is replaced by repulsion at distances $r < M/2$.

The determinant of metric (24) is equal to

$$\det g_{\alpha\beta} = - \left(1 - \frac{M}{2r}\right)^2 \left(1 + \frac{M}{2r}\right)^{10} r^4 \sin^2 \theta. \quad (25)$$

Consequently, the metric is degenerate on the sphere of radius

$$r_* := \frac{M}{2}, \quad (26)$$

which corresponds to the horizon $\rho = 2M$ for the Schwarzschild solution and on the z -axis ($\theta = 0, \pi$). Degeneracy on the z -axis is related to the spherical coordinate system.

It is well known that all geometric invariants for the Schwarzschild metric constructed from the curvature tensor have no singularities on the horizon.

When radius r decreases from infinity to the critical value $r_* := M/2$, the Schwarzschild radial coordinate decreases from ∞ to the horizon $\rho_s := 2M$. Afterwards the radius ρ increases from ρ_s to ∞ as radius r decreases from r_* to zero. We see that point massive particle is located at that site where the Schwarzschild radial coordinate equals infinity. Thus two copies of the Schwarzschild metric outside horizon $2M < \rho < \infty$ are mapped on two distinct domains in \mathbb{R}^3 : $0 < r < r_*$ and $r_* < r < \infty$. On the sphere $r = r_*$ they are smoothly glued together. The spatial part of the metric is not degenerate here. Note that area of a sphere surrounding a particle tends to infinity as it gets closer to it. It is related to the fact that components of the spatial part of the metric (24) diverge for $r \rightarrow 0$.

One can easily calculate the asymptotic of the zeroth component of the metric (24)

$$g_{00} \approx 1 - \frac{2M}{r}, \quad r \rightarrow \infty.$$

It is the same as for the Schwarzschild metric. This is not surprising because $\rho \rightarrow r$ for $r \rightarrow \infty$. It means that there is no problem with Newton's gravitational law for metric (24) because it is defined by the asymptotic at $r \rightarrow \infty$.

4 Point mass in General Relativity

Let us consider topologically trivial manifold $\mathbb{M} \approx \mathbb{R}^4$ (space-time) with Cartesian coordinates x^α , $\alpha = 0, 1, 2, 3$, and metric $g_{\alpha\beta}(x)$ of Lorentzian signature $\text{sign } g_{\alpha\beta} = (+ - - -)$. We denote the world line of a point particle by $\{q^\alpha(\tau)\}$ where τ is a parameter along the world line. In General Relativity, a point particle of mass M is described by the following action

$$S = \frac{1}{16\pi} \int dx \sqrt{|g|} R - M \int d\tau \sqrt{\dot{q}^\alpha \dot{q}^\beta g_{\alpha\beta}}, \quad (27)$$

where $g := \det g_{\alpha\beta}$, R is the scalar curvature, and $\dot{q} := dq/d\tau$.

Variations of action (27) with respect to metric components $g_{\alpha\beta}$ and coordinates of a particle q^α yield Einstein's equations of motion and equations for extremals (geodesics):

$$R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = -\frac{1}{2}T^{\alpha\beta}, \quad (28)$$

$$\left(\ddot{q}^\alpha + \Gamma_{\beta\gamma}^\alpha|_{x=q} \dot{q}^\beta \dot{q}^\gamma\right) g_{\alpha\delta} = 0, \quad (29)$$

where

$$T^{\alpha\beta} = \frac{16\pi M \dot{q}^\alpha \dot{q}^\beta}{\sqrt{|g|} \dot{q}^0} \delta(\mathbf{x} - \mathbf{q}) \quad (30)$$

is the particle energy-momentum tensor, $\Gamma_{\beta\gamma}^\alpha$ are Christoffel's symbols for metric $g_{\alpha\beta}$, and

$$\delta(\mathbf{x} - \mathbf{q}) := \delta(x^1 - q^1) \delta(x^2 - q^2) \delta(x^3 - q^3)$$

is the three-dimensional δ -function on sections $x^0 = \text{const}$. We assume that for particle trajectory $\dot{q}^0 \neq 0$.

Usually, Einstein's equations are written with lower covariant indices. We are seeking solutions of Einstein's equations for metric which components may have zeroes and singularities. Therefore equations with upper contravariant and lower covariant indices can be inequivalent.

We have solved equation (5) for three-dimensional scalar curvature with δ -function on the right hand side. However it is not clear at all what relation this solution has to system of equations (28)–(29).

To analyze system of equations (28)–(29), we rewrite it in the Hamiltonian form. To this end, the ADM parameterization of the metric is used

$$g_{\alpha\beta} = \begin{pmatrix} N^2 + N^\rho N_\rho & N_\nu \\ N_\mu & g_{\mu\nu} \end{pmatrix}, \quad (31)$$

where $g_{\mu\nu}$ is the metric on space like sections of the space-time $x^0 = \text{const}$. In this parameterization, 4 functions: the lapse function $N(x)$ and shift functions $N_\mu(x)$, are introduced instead of 4 components g_{00} and $g_{0\mu}$ of the metric. Here we use notation $N^\rho := \hat{g}^{\rho\mu} N_\mu$ where $\hat{g}^{\rho\mu}$ is the 3×3 -matrix which is inverse to $g_{\mu\nu}$:

$$\hat{g}^{\rho\mu} g_{\mu\nu} = \delta_\nu^\rho.$$

In what follows, raising of space indexes which are denoted by Greek letters from the middle of the alphabet, $\mu, \nu, \dots = 1, 2, 3$, is performed using the inverse three-dimensional metric marked with a hat. We assume that all sections $x^0 = \text{const}$ are space like, and therefore the metric $g_{\mu\nu}$ is negative definite.

Let $p^{\mu\nu}$ and p_α be momenta conjugate to generalized coordinates $g_{\mu\nu}$ and q^α . The action for a point particle is invariant with respect to reparameterization of the world line. To simplify formulae, we fix the gauge $\tau = q^0$. Then Einstein's equations (28) in the Hamiltonian form reduce to constraint and dynamical equations. Constraints have the form

$$H_\perp = \frac{1}{\hat{e}} (p^{\mu\nu} p_{\mu\nu} - p^2) - \hat{e} \hat{R} + \sqrt{M^2 + \hat{p}^2} \delta(\mathbf{x} - \mathbf{q}) = 0, \quad (32)$$

$$H_\mu = -2\hat{\nabla}_\nu p^\nu{}_\mu - p_\mu \delta(\mathbf{x} - \mathbf{q}) = 0, \quad (33)$$

where $\hat{p}^2 := \hat{g}^{\mu\nu} p_\mu p_\nu$. Constraints (32) and (33) are called dynamical and kinematical, respectively. Dynamical Hamiltonian Einstein's equations are

$$\dot{g}_{\mu\nu} = \frac{2N}{\hat{e}} p_{\mu\nu} - \frac{N}{\hat{e}} g_{\mu\nu} p + \hat{\nabla}_\mu N_\nu + \hat{\nabla}_\nu N_\mu, \quad (34)$$

$$\begin{aligned} \dot{p}^{\mu\nu} = & \frac{N}{2\hat{e}} \hat{g}^{\mu\nu} \left(p^{\rho\sigma} p_{\rho\sigma} - \frac{1}{2} p^2 \right) - \frac{2N}{\hat{e}} \left(p^{\mu\rho} p^\nu{}_\rho - \frac{1}{2} p^{\mu\nu} p \right) + \hat{e} (\hat{\Delta} N \hat{g}^{\mu\nu} - \hat{\nabla}^\mu \hat{\nabla}^\nu N) - \\ & - \hat{e} N \left(\hat{R}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \hat{R} \right) - p^{\mu\rho} \hat{\nabla}_\rho N^\nu - p^{\nu\rho} \hat{\nabla}_\rho N^\mu + \hat{\nabla}_\rho (N^\rho p^{\mu\nu}) - \frac{N p^\mu p^\nu}{2\sqrt{M^2 + \hat{p}^2}} \delta(\mathbf{x} - \mathbf{q}), \end{aligned} \quad (35)$$

where $\hat{\Delta} := \hat{\nabla}^\mu \hat{\nabla}_\mu$ is the three-dimensional Laplace–Beltrami operator and $p^\mu := \hat{g}^{\mu\nu} p_\nu$. Hamiltonian equations for geodesics have the form

$$\dot{q}^\mu = - \frac{N}{\sqrt{M^2 + \hat{p}^2}} \Big|_{\mathbf{x}=\mathbf{q}} p^\mu - N^\mu|_{\mathbf{x}=\mathbf{q}}, \quad (36)$$

$$\dot{p}_\mu = - \partial_\mu \left[N \sqrt{M^2 + \hat{p}^2} - N^\nu p_\nu \right]_{\mathbf{x}=\mathbf{q}}. \quad (37)$$

Transition from Lagrangian equations of motion (28), (29) to Hamiltonian (32)–(37) is complicated. Hamiltonian formulation of General Relativity was given in [10, 4]. For General Relativity and point particle it can be found i.e. in [11]. Combined Hamiltonian system of equations of motion for a point particle in General Relativity was considered in [3, 12].

We are seeking spherically symmetric static solution of the system of equations of motion (32)–(37). In this case, shift functions are zero, $N_\mu = 0$. We choose the Weyl flat gauge for the spatial metric

$$g_{\mu\nu} = -f^2 \delta_{\mu\nu}.$$

Assume also that the particle is located at the origin of the coordinate system:

$$q^0 = x^0 = \tau, \quad q^\mu = 0, \quad \mu = 1, 2, 3.$$

In the static case, all momenta vanish: $p^{\mu\nu} = 0$ and $p_\mu = 0$. Derivatives $\dot{g}_{\mu\nu} = 0$ and $\dot{q}^\mu = 0$ are also zero. Thus there are two unknown functions in this case which depend only on radius in spherical coordinates: $N(r)$ and $f(r)$.

If the above assumptions are fulfilled then equations (33), (34), and (36) are identically satisfied. Equations (32), (35), and (37) take the form

$$-\hat{e} \hat{R} + 16\pi M \delta(\mathbf{x}) = 0, \quad (38)$$

$$\hat{e} \left(\hat{\Delta} N \hat{g}^{\mu\nu} - \hat{\nabla}^\mu \hat{\nabla}^\nu N \right) - \hat{e} N \left(\hat{R}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \hat{R} \right) = 0, \quad (39)$$

$$\partial_\mu N|_{\mathbf{x}=0} = 0, \quad (40)$$

where equation (37) we divided by M .

It is important that equation (39) can not be divided by \hat{e} , and indices μ, ν can not be lowered because metric have singularity at $r = 0$, and the Ricci tensor and scalar curvature contain the δ -function.

Equation (38) was solved in section 2. The solution is given by equation (11). The difference in the sign in equation (38) arises because the spatial metric $g_{\mu\nu}$ is now negative definite, and changing sign of the metric results in changing sign of the scalar curvature.

From the Schwarzschild metric in the form (24), we deduce that solution of equation (39) for the lapse function is

$$N = \frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}}. \quad (41)$$

The lapse function is positive for $r > M/2$ and negative for $0 < r < M/2$. It has finite limit at zero,

$$\lim_{r \rightarrow 0} N = -1,$$

and is infinitely differentiable for $r > 0$. Therefore the lapse function is locally summable and lies in $\mathcal{D}'(\mathbb{R}^3)$.

Theorem 4.1. *Lapse function (41) satisfies equation (39) in \mathbb{R}^3 in a generalized sense.*

Proof. One can easily verify that the Laplacian of the lapse function for $r > 0$ is equal to zero,

$$\hat{\Delta}N = \frac{1}{\hat{e}} \partial_\mu (\hat{e} \hat{g}^{\mu\nu} \partial_\nu N) = 0.$$

Function N is continuous, and its derivatives have discontinuities at zero. Since $N \in \mathcal{D}'(\mathbb{R}^3)$ then the weak derivatives are defined everywhere, and equality

$$\hat{\Delta}N = 0,$$

is fulfilled in a generalized sense because the lapse function has finite limit at zero. This equality can be multiplied by $\hat{e}g^{\mu\nu}$ because near zero

$$\hat{e} \sim r^{-4}, \quad g^{\mu\nu} \sim r^4. \quad (42)$$

Thus the first term on the left hand side of equation (39) is equal to zero in a generalized sense.

One can verify that for $r > 0$ the following equality holds

$$\hat{e} \hat{\nabla}^\mu \hat{\nabla}^\nu N + \hat{e} N \left(\hat{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \hat{R} \right) = 0.$$

Hence we are left only with δ -functions which are contained in the Ricci tensor and scalar curvature. They do not cancel, and equation (39) takes the form

$$\frac{1}{4} g^{\mu\nu} N \hat{e} \hat{R} = 4\pi M g^{\mu\nu} N \delta(\mathbf{x}) = 0.$$

This equation is fulfilled because

$$g^{\mu\nu} N \sim \delta^{\mu\nu} r^4, \quad r^4 \delta(\mathbf{x}) = 0$$

near zero. □

Thus we proved that the Schwarzschild metric in isotropic coordinates (24) satisfies Einstein's equations (28) for point massive particle in a generalized sense.

We are left only with equation (40) for the point particle. Since

$$\partial_r N = \frac{M}{r^2} \frac{1}{\left(1 + \frac{M}{2r}\right)^2},$$

it is not satisfied. At this point, we invoke physical arguments. The right hand side of equation (37) is the force acting on the particle from the gravitational field which is produced by the particle itself. Therefore equation (37) is understood after averaging over a sphere surrounding the particle. Then it is fulfilled because of the spherical symmetry. In other words, the particle's own gravitational field does not act on it.

The same situation happens in Newton's gravity and electrodynamics. For example, for gravitational potential $\varphi = -1/r$ in Newton's theory, the following equality holds

$$-\partial_\mu \frac{1}{r} = \frac{x^\mu}{r^3},$$

and equations of motion are not satisfied. Nevertheless they are fulfilled after averaging over a sphere.

5 Geodesics

To give physical meaning for metric (24), it is necessary to analyze the behavior of geodesics or extremals (lines of extremal length). We assume in General Relativity that test particles move along geodesics. To this end we compute Christoffel's symbols (1). Straightforward calculations show that only nine independent components differ from zero:

$$\begin{aligned} \Gamma_{00}^r &= \frac{1 - \frac{M}{2r}}{\left(1 + \frac{M}{2r}\right)^7} \frac{M}{r^2}, & \Gamma_{0r}^0 &= \frac{1}{\left(1 + \frac{M}{2r}\right) \left(1 - \frac{M}{2r}\right)} \frac{M}{r^2}, \\ \Gamma_{rr}^r &= -\frac{1}{1 - \frac{M}{2r}} \frac{M}{r^2}, & \Gamma_{r\theta}^\theta &= \Gamma_{r\varphi}^\varphi = \frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}} \frac{1}{r}, \\ \Gamma_{\theta\theta}^r &= -\frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}} r, & \Gamma_{\theta\varphi}^\varphi &= \frac{\cos \theta}{\sin \theta}, \\ \Gamma_{\varphi\varphi}^r &= -\frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}} r \sin^2 \theta, & \Gamma_{\varphi\varphi}^\theta &= -\sin \theta \cos \theta. \end{aligned} \tag{43}$$

Therefore equations for geodesics $\{x^\alpha(\tau)\}$, $\alpha = 0, 1, 2, 3$,

$$\ddot{x}^\alpha = -\Gamma_{\beta\gamma}^\alpha \dot{x}^\beta \dot{x}^\gamma,$$

where $\dot{x}^\alpha := dx^\alpha/d\tau$ is the differentiation with respect to canonical parameter take the form

$$\begin{aligned}
\ddot{t} &= -\frac{2}{\left(1 + \frac{M}{2r}\right)\left(1 - \frac{M}{2r}\right)} \frac{M}{r^2} \dot{t}\dot{r}, \\
\ddot{r} &= -\frac{1 - \frac{M}{2r}}{\left(1 + \frac{M}{2r}\right)^7} \frac{M}{r^2} \dot{t}^2 + \frac{1}{1 + \frac{M}{2r}} \frac{M}{r^2} \dot{r}^2 + \frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}} r(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2), \\
\ddot{\theta} &= -\frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}} \frac{2}{r} \dot{r}\dot{\theta} + \sin\theta \cos\theta \dot{\varphi}^2, \\
\ddot{\varphi} &= -\frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}} \frac{2}{r} \dot{r}\dot{\varphi} - 2\frac{\cos\theta}{\sin\theta} \dot{\theta}\dot{\varphi}.
\end{aligned} \tag{44}$$

Equations for geodesics always have the integral of motion

$$C_0 = \dot{x}^\alpha \dot{x}^\beta g_{\alpha\beta} = \dot{t}^2 \left(\frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}} \right)^2 - \left(1 + \frac{M}{2r} \right)^4 [\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2)]. \tag{45}$$

Its geometric meaning is that the length of a tangent vector is conserved along geodesic. Moreover, each Killing vector $K = K^\alpha \partial_\alpha$ yields the integral of motion

$$C = K^\alpha \dot{x}^\beta g_{\alpha\beta} = \text{const.}$$

The obtained solution is static which means that $K_1 = \partial_0$ is the Killing vector. There are three Killing vectors corresponding to spherical symmetry from which only two are independent. Thus there are three integrals of motion for geodesics in metric (24) related to the symmetry of the metric. Existence of four integrals of motion for system of equations (44) allows to analyze qualitatively the behavior of all geodesics. We analyze only the simplest.

For Killing vector fields $K_1 = \partial_0$ and $K_2 = \partial_\varphi$, we have integrals of motion:

$$C_1 = \dot{t} \left(\frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}} \right)^2, \tag{46}$$

$$C_2 = -\dot{\varphi} \left(1 + \frac{M}{2r} \right)^4 r^2 \sin^2\theta. \tag{47}$$

Differentiating $C_{0,1,2}$ with respect to canonical parameter and using equations of motion (44) one can prove that $C_{0,1,2}$ are integrals of motion, indeed.

Using the freedom in the choice of a canonical parameter, we put $C_1 = 1$ for $r > r_* := M/2$. Then at large distances, $r \gg M$, where the gravitational field is small conservation law (46) takes the form $\dot{t} = 1$ at zeroth order. This means that at large distances the canonical parameter τ coincides with observational time t . This gives physical meaning to constant C_1 .

At large distances in the main approximation with respect M/r and for $\dot{t} = 1$, second equation for radius in (44) reduces to

$$\ddot{r} = -\frac{M}{r^2} + \frac{M}{r^2} \dot{r}^2 + r(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2).$$

If we require in addition smallness of velocities $\dot{r} \ll 0$, then the second term on the right hand side can be neglected as compared with the first term. Thus at large distances and small velocities,

$$\frac{M}{r} \sim \epsilon, \quad \dot{r} \sim \epsilon, \quad \epsilon \ll 1,$$

equations for geodesics in General Relativity can be written in the form

$$\ddot{r} = -\frac{M}{r^2} + r(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2), \quad (48)$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} + \sin \theta \cos \theta \dot{\varphi}^2, \quad (49)$$

$$\ddot{\varphi} = -\frac{2\dot{r}\dot{\varphi}}{r} - 2\frac{\cos \theta}{\sin \theta} \dot{\theta} \dot{\varphi}, \quad (50)$$

first equation in (44) being satisfied in the main approximation. In this form, equations for geodesics can be compared with equations of motion for a point particle in classical Newton's theory.

Newtonian limit. In Newton's gravity theory, a point particle of unit mass moves in three-dimensional Euclidean space in the potential field $U = -M/r$. Its motion along trajectory $\{x^\mu(t)\}$ is defined by the following action functional

$$S = \int dt L := \int dt \left(\frac{v^2}{2} + \frac{M}{r} \right) = \int dt \left(\frac{\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2)}{2} + \frac{M}{r} \right), \quad (51)$$

where v is the velocity of the particle which is written in spherical coordinate system. Variation of this functional with respect to $r(t)$, $\theta(t)$, and $\varphi(t)$ results exactly to equations of motion (48)–(50) where dot denotes differentiation with respect to time t .

Invariance of action (51) with respect to time translation $t \mapsto t + \text{const}$, according to the Noether theorem, leads to the energy conservation

$$E = \frac{v^2}{2} - \frac{M}{r} = \frac{\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2)}{2} - \frac{M}{r} = \text{const}. \quad (52)$$

Invariance of action (51) with respect to angle shift $\varphi \mapsto \varphi + \text{const}$ leads to conservation of the angular momentum projection on z axis:

$$L = -\frac{\partial L}{\partial \dot{\varphi}} = -r^2 \sin^2 \theta \dot{\varphi} = \text{const}. \quad (53)$$

Similarly, other angular momentum projections on x and y axes are conserved though their expressions in spherical coordinates look more complicated.

Since the angular momentum is conserved, a particle moves on the plane which goes through the origin and the vector of initial velocity. Therefore without loss of generality, we can restrict motion to a single plane, say $\theta = \pi/2$.

Thus at distances much larger then r_* which correspond to the Schwarzschild radius $\rho_s := 2M$ and velocities small as compared to the velocity of light, General Relativity reproduces equations of motion for a point particle in Newton's theory.

Note that the Schwarzschild radius is small. Substitution of the numerical values yields the following values: for the Sun and Earth the Schwarzschild radii are equal to 3 km and 1 cm, respectively.

Comparison of conservation laws (55) and (52), and also (47) and (53) tells us that integration constants C_0 and C_2 for geodesic equations are analogs of energy and angular momentum conservation in General Relativity, respectively. \square

As the consequence of angular momentum conservation (C_2) we restrict ourselves to geodesics in the plane $\theta = \pi/2$. Then $\dot{\theta} = 0$, and equations (44) are simplified:

$$\begin{aligned}\ddot{t} &= -\frac{2}{\left(1 + \frac{M}{2r}\right)\left(1 - \frac{M}{2r}\right)} \frac{M}{r^2} \dot{t} \dot{r}, \\ \ddot{r} &= -\frac{1 - \frac{M}{2r}}{\left(1 + \frac{M}{2r}\right)^7} \frac{M}{r^2} \dot{t}^2 + \frac{1}{1 + \frac{M}{2r}} \frac{M}{r^2} \dot{r}^2 + \frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}} r \dot{\varphi}^2, \\ \ddot{\varphi} &= -\frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}} \frac{2}{r} \dot{r} \dot{\varphi}\end{aligned}\tag{54}$$

Integrals of motion (55) and (47) become slightly simpler:

$$C_0 = \dot{t}^2 \left(\frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}} \right)^2 - \left(1 + \frac{M}{2r} \right)^4 [r^2 + r^2 \dot{\varphi}^2],\tag{55}$$

$$C_2 = -\dot{\varphi} \left(1 + \frac{M}{2r} \right)^4 r^2.\tag{56}$$

Integral of motion (46) remains the same.

For further analysis of geodesics, we put $C_1 = 1$. This means that for infinitely distant observer the canonical parameter along geodesics coincides with the observed time $\tau \simeq t$.

Circular geodesics. If a test particle moves along a circle $r = \text{const}$, then first equation in (54) takes the form $\dot{t} = 0$. Therefore without loss of generality we can identify the observed time with canonical parameter, $t = \tau$. Then second equation in (54) defines the angular velocity:

$$\dot{\varphi}^2 = \frac{1}{\left(1 + \frac{M}{2r}\right)^6} \frac{M}{r^3} = \text{const}.\tag{57}$$

At large distances, $r \gg M$, the obtained angular velocity of rotations coincides with that in Newton's mechanics:

$$\dot{\varphi}^2 = \frac{M}{r^3}.$$

Third equation in (54) for circular geodesics is automatically fulfilled.

As the radius r decreases from infinity to the critical value r_* the angular velocity of rotations increases from zero to its maximal value at $r = r_*$. Then the angular velocity of rotations starts decreasing which means that attraction changes to repulsion. Near the particle it decreases to zero

$$\lim_{r \rightarrow 0} \dot{\varphi}^2 = 0.$$

This means that near the origin of the coordinate system a test particle does not feel any gravitational force.

Lightlike radial geodesics. For lightlike geodesics, $C_0 = 0$. Besides, for radial motion of light, $\dot{\varphi} = 0$. In this case, conservation law (55) allows to find

$$\dot{t} = \pm \frac{\left(1 + \frac{M}{2r}\right)^3}{1 - \frac{M}{2r}} \dot{r}.\tag{58}$$

On the other hand, conservation law (46) for $C_1 = 1$ leads to equality

$$\dot{t} = \left(\frac{1 + \frac{M}{2r}}{1 - \frac{M}{2r}} \right)^2. \quad (59)$$

The obtained equations allow to find the velocity

$$\dot{r} = \pm \frac{1}{1 - \frac{M^2}{4r^2}}. \quad (60)$$

These equations are easily integrated:

$$\pm \tau + \text{const} = r + \frac{M^2}{4r}. \quad (61)$$

As the consequence, the points $r = 0$ and $r = \infty$ are complete for radial lightlike geodesics. That is radially falling light never reaches a point particle of mass M . This is what is to be expected because the domain near the particle $r \ll M$ is diffeomorphic to the external Schwarzschild solution for $\rho \gg M$ which is complete as $\rho \rightarrow \infty$, this limit corresponding to $r \rightarrow 0$. In particular, no timelike curve can reach the origin of the coordinate system $r = 0$ at finite proper time.

Radial equation (54) with equalities (59) and (60) yield equation for the radius

$$\ddot{r} = - \frac{1}{\left(1 - \frac{M^2}{4r^2}\right)^3} \frac{M^2}{2r^3}, \quad (62)$$

where acceleration stands on the left hand side. At large distances $r > r_*$ acceleration is negative, and this corresponds to attraction. At small distances $r < r_*$ acceleration is positive, and this means repulsion. This consequence of Einstein's equations has nothing in common with Newtonian gravity where we have only attraction.

At large distances as well as near the critical radius and the origin of the coordinate system velocity and acceleration have the following behavior:

$$\begin{array}{lll} r \rightarrow \infty : & \dot{r} \rightarrow \pm 1, & \ddot{r} \rightarrow 0, \\ r \rightarrow r_* : & \dot{r} \rightarrow \pm \infty, & \ddot{r} \rightarrow \mp \infty, \\ r \rightarrow 0 : & \dot{r} \rightarrow 0, & \ddot{r} \rightarrow 0. \end{array}$$

That is, at infinity and near the origin, light moves without acceleration. When crossing the critical sphere $r = r_*$ the acceleration changes its sign which means that attraction is replaced by repulsion.

6 Conclusion

In the present paper we prove that the Schwarzschild in isotropic coordinates satisfies the system of Einstein's and geodesic equations for a point massive particle. Nontrivial energy-momentum tensor appears on the right hand side of Einstein's equations which is proportional to three-dimensional δ -function. Rewriting the Schwarzschild solution in isotropic coordinates is by itself a student's problem. The result of the paper is that we

attribute mathematical meaning for Einstein's equations with δ -type source and prove that the Schwarzschild solution in isotropic coordinates describes gravitational field of a point massive particle.

The mass M appears in the Schwarzschild solution as an integration constant and may be arbitrary. In our approach, the mass M of a point particle enters the action from the very beginning.

The obtained solution is isometric to Einstein–Rosen bridge [13]. In the original paper the bridge was attributed to an elementary particle which in our notations correspond to the critical sphere $r = r_*$. The space-time is described by two sheets. We have shown that the particle described by a δ -function is located not at $r = r_*$, but at geodesically complete “infinity” of one of the sheets corresponding to $r = 0$. Geodesically complete infinity of the second sheet lies at infinity $r = \infty$ and corresponds to asymptotically flat space-time. Both sheets are glued together at the critical sphere $r = r_*$ where metric becomes degenerate and gravitational attraction at large distances is replaced by repulsion at small distances. This effect is of primary importance and makes us to reconsider our approach to many gravitational phenomena.

At present, the Einstein–Rosen bridge has another interpretation. Two sheets are considered as two different universes which are connected by a worm hole at $r = r_*$. It was noted recently [14] that nontrivial energy-momentum tensor appears on the right hand side of Einstein's equations at r_* . In our approach, the right hand side of Einstein's equations is equal to zero at $r = r_*$. The difference comes from choosing the solution to equation $N^2 = g_{00}$. For a given metric we chose solution (41) which is positive for $r > r_*$ and negative at $r < r_*$. The other solution $|N|$ is chosen in [14], the modulus sign resulting in the appearance of the singularity on the right hand side of Einstein's equations at $r = r_*$.

The obtained space-time is topologically four-dimensional Euclidean space with removed world line of the particle $r = 0$. This space-time is geodesically complete.

The effect of the transformation of gravitational attraction into repulsion is the straightforward consequence of Einstein's equations and leads to many questions. For example, how can a black hole be formed if particles cannot come close one to another? One can make a lot of speculations at this point but we do not consider them here.

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